Gravitational Field of Massive Point Particle in General Relativity

P. P. Fiziev*

Department of Theoretical Physics,
Faculty of Physics, Sofia University,
5 James Bourchier Boulevard, Sofia 1164, Bulgaria.
and
The Abdus Salam International Centre for Theoretical Physics,
Strada Costiera 11, 34014 Trieste, Italy.

Utilizing various gauges of the radial coordinate we give a description of static spherically symmetric space-times with point singularity at the center and vacuum outside the singularity. We show that in general relativity (GR) there exist a two-parameters family of such solutions to the Einstein equations which are physically distinguishable but only some of them describe the gravitational field of a single massive point particle with nonzero bare mass M_0. In particular, we show that the widespread Hilbert's form of Schwarzschild solution, which depends only on the Keplerian mass $M < M_0$, does not solve the Einstein equations with a massive point particle's stress-energy tensor as a source. Novel normal coordinates for the field and a new physical class of gauges are proposed, in this way achieving a correct description of a point mass source in GR. We also introduce a gravitational mass defect of a point particle and determine the dependence of the solutions on this mass defect. The result can be described as a change of the Newton potential $\varphi_N = -G_N M/r$ to a modified one: $\varphi_G = -G_N M/\left(r + G_N M/c^2 \ln \frac{M_0}{M}\right)$ and a corresponding modification of the four-interval. In addition we give invariant characteristics of the physically and geometrically different classes of spherically symmetric static space-times created by one point mass. These space-times are analytic manifolds with a definite singularity at the place of the matter particle.
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I.   INTRODUCTION

The Einstein's equations:

$$G^\mu_\nu = \kappa T^\mu_\nu \qquad (1)$$

determine the solution of a given physical problem up to four arbitrary functions, i.e., up to a choice of coordinates. This reflects the well known fact that GR is a gauge theory.

According to the standard textbooks [1] the fixing of the gauge in GR in a *holonomic* frame is represented by a proper choice of the quantities

$$\bar{\Gamma}_\mu = -\frac{1}{\sqrt{|g|}} g_{\mu\nu} \partial_\lambda \left(\sqrt{|g|} g^{\lambda\nu}\right), \qquad (2)$$

which emerge when one expresses the 4D d'Alembert operator in the form $g^{\mu\nu} \nabla_\mu \nabla_\nu = g^{\mu\nu} \left(\partial_\mu \partial_\nu - \bar{\Gamma}_\mu \partial_\nu\right)$.

We shall call the change of the gauge fixing expressions (2), *without* any preliminary conditions on the analytical behavior of the used functions, a gauge transformations in a *broad* sense. This way we essentially expand the class of admissible gauge transformations in GR. Of course this will alter some of the well known *mathematical* results in the commonly used mathematical scheme of GR. We think that a careful analysis of such wider framework for the gauge transformations in GR can help us clarify some long standing *physical* problems. After all, in the

physical applications the mathematical constructions are to reflect in an adequate way the properties of the real physical objects.

It is well known that in the gauge theories we may have different solutions with the same symmetry in the base space. Such solutions belong to different gauge sectors, owning different geometrical, topological and physical properties. For a given solution there exists a class of *regular* gauges, which alter the form of the solution without changing these essential properties. In contrast, by performing a *singular* gauge transformation one can change both the geometrical, topological and physical properties of the solution, making a transition to another gauge sector.

In geometrical sense the different solutions in GR define different 4D pseudo-Riemannian space-time manifolds $\mathcal{M}^{(1,3)}\{g_{\mu\nu}(x)\}$. A subtle point in GR formalism is that transitions from a given physical solution to an essentially different one, can be represented as a "change of coordinates". This is possible because from gauge theory point of view the choice of space-time coordinates in GR means simultaneously two different things: choosing a gauge sector in which the solution lives and at the same time fixing the (regular) gauge in this sector. While the latter solves an inessential local gauge fixing problem, the first fixes the essential global properties of the solution.

One can understand better this peculiarity of GR in the framework of its modern differential geometrical description as a gauge theory, using the principal frame bundle $F\left(\mathcal{M}^{(1,3)}\{g_{\mu\nu}(x)\}\right)$. (See, for example, the second edition of the monograph [2], [3], and the references therein.) The change of coordinates on the base

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*E-mail: fiziev@phys.uni-sofia.bg

space $\mathcal{M}^{(1,3)}\{g_{\mu\nu}(x)\}$ induces *automatically* a nontrivial change of the frames on the fibre of frames. Singular coordinate transformations may produce a change of the gauge sector of the solution, because they may change the topology of the frame bundle, adding new singular points and(or) singular sub-manifolds, or removing some of the existing ones. For example, such transformations can alter the fundamental group of the base space $\mathcal{M}^{(1,3)}\{g_{\mu\nu}(x)\}$, the holonomy group of the affine connection on $F\left(\mathcal{M}^{(1,3)}\{g_{\mu\nu}(x)\}\right)$, etc. Thus one sees that coordinate changes in broad sense are more than a pure alteration of the labels of space-time points.

If one works in the framework of the theory of *smooth real* manifolds, ignoring the analytical properties of the solutions in the complex domain, one is generally allowed to change the gauge without an alteration of the physical problem in the *real* domain, i.e. without change of the boundary conditions, as well as without introduction of new singular points, or change of the character of the existing ones. Such special type of *regular* gauge transformations in GR describe the diffeomorphisms of the *real* manifold $\mathcal{M}^{(1,3)}\{g_{\mu\nu}(x)\}$. This manifold is already fixed by the initial choice of the gauge. Hence, the *real* manifold $\mathcal{M}^{(1,3)}\{g_{\mu\nu}(x)\}$ is actually described by a class of equivalent gauges, which correspond to all diffeomorphisms of this manifold and are related with regular gauge transformations.

The transitions between some specific real manifolds $\mathcal{M}^{(3)}\{-g_{mn}(\mathbf{r})\}$, which are not diffeomorphic, can be produced by the use of proper *singular* gauge transformations. If we start from an everywhere smooth manifold, after a singular transformation we will have a new manifold with some singularities, which are to describe real physical phenomena in the problem at hand. The inclusion of such singular transformations in our consideration yields the necessity to talk about gauge transformations *in a broad sense*. They are excluded from present-day standard considerations by the commonly used assumption, that in GR one has to allow *only* diffeomorphic mappings.

Similar singular transformations are well known in gauge theories of other fundamental physical interactions: electromagnetic interactions, electroweak interactions, chromodynamics. For example, in the gauge theories singular gauge transformations describe transitions between solutions in topologically different gauge sectors. Singular gauge transformations are used in the theory of Dirac monopole, vortex solutions, t'Hooft-Poliakov monopoles, Yang-Mills instantons, etc. See, for example, [3, 4] and the references therein.

The simplest example is the singular gauge transformation of the 3D vector potential: $\mathbf{A}(\mathbf{r}) \rightarrow \mathbf{A}(\mathbf{r}) + \nabla\varphi(\mathbf{r})$ in electrodynamics, defined in Cartesian coordinates $\{x, y, z\}$ by the singular gauge function $\varphi = \alpha \arctan(y/x)$ ($\alpha = const$). Suppose that before the transformation we have had a 3D space $\mathcal{M}^{(3)}\{-g_{mn}(\mathbf{r}) = \delta_{mn}\} = \mathcal{R}^{(3)}\{\delta_{mn}\}$. Then this singular gauge transformation removes the whole axes OZ out of the Euclidean

3D space $\mathcal{R}^{(3)}\{\delta_{\mu\nu}\}$, changing the topology of this part of the base space. As a result the quantity $\oint_C d\mathbf{r}\mathbf{A}(\mathbf{r})$, which is gauge invariant under regular gauge transformations, now changes its value to $\oint_C d\mathbf{r}\mathbf{A}(\mathbf{r}) + 2\pi N\alpha$, where N is the winding number of the cycle C around the axes OZ. Under such singular gauge transformation the solution of some initial physical problem will be transformed onto a solution of a completely different problem.

At present the role of singular gauge transformations in the above physical theories is well understood.

In contrast, we still don't have systematic study of the classes of physically, geometrically and topologically different solutions in GR, created by singular gauge transformations, even in the simple case of static spherically symmetric space-times with only one point singularity, although the first solution of this type was discovered first by Schwarzschild nearly 90 years ago [5]. Moreover, in GR at present there is no clear understanding both of the above gauge problem and of its physical significance.

Here we present some initial steps toward the clarification of the role of different GR gauges in broad sense for spherically symmetric static space-times with point singularity at the center of symmetry and vacuum outside this singularity.

II. STATIC SPHERICALLY SYMMETRIC SPACE-TIMES WITH POINT SOURCE OF GRAVITY

The static point particle with bare mechanical rest mass M_0 can be treated as a 3D entity. Its proper frame of reference is most suitable for description of the *static* space-time with this single particle in it. We prefer to present the problem of a point source of gravity in GR as an 1D mathematical problem, considering the dependence of the corresponding functions on the only essential variable – the radial variable r. This can be achieved in the following way.

The spherical symmetry of the 3D space reflects adequately the point character of the source of gravity. A real spherically symmetric 3D Riemannian space $\mathcal{M}^{(3)}\{-g_{mn}(\mathbf{r})\} \subset \mathcal{M}^{(1,3)}\{g_{\mu\nu}(x)\}$ can be described using standard spherical coordinates $r \in [0, \infty)$, $\theta \in [0, \pi)$, $\phi \in [0, 2\pi)$. Then

$$\mathbf{r} = \{x^1, x^2, x^3\} = \{r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta\}.$$

The physical and geometrical meaning of the radial coordinate r is not defined by the spherical symmetry of the problem and is unknown *a priori* [6]. The only clear thing is that its value $r = 0$ corresponds to the center of the symmetry, where one must place the physical source of the gravitational field. In the present article we assume that there do not exist other sources of gravitational field outside the center of symmetry.

There exists unambiguous choice of a global time t due to the requirement to use a static metric. In proper units

it yields the familiar form of the space-time interval [9]:

$$ds^2 = g_{tt}(r)\, dt^2 + g_{rr}(r)\, dr^2 - \rho(r)^2 (d\theta^2 + \sin^2\theta\, d\phi^2) \quad (3)$$

with unknown functions $g_{tt}(r) > 0$, $g_{rr}(r) < 0$, $\rho(r)$.

In contrast to the variable r, the quantity ρ has a clear geometrical and physical meaning: It is well known that ρ defines the area $A_\rho = 4\pi\rho^2$ of a centered at $r = 0$ sphere with "a radius" ρ and the length of a big circle on it $l_\rho = 2\pi\rho$. One can refer to this quantity as an *"area radius"*, or as an optical *"luminosity distance"*, because the luminosity of distant physical objects is reciprocal to A_ρ.

One has to stress that in the proper frame of reference of the point particle this quantity does not measure the *real* geometrical distances in the corresponding curved space-time. In contrast, if ρ_{fixed} is some arbitrary fixed value of the luminosity distance, the expression $(\rho_2 - \rho_1)/(1 - 2M/\rho_{fixed})^{1/2}$ measures the 3D geometrical distance between the geometrical points 2 and 1 on a radial geodesic line in the frame of free falling clocks [7]. Nevertheless, even in this frame the absolute value of the variable ρ remains not fixed by the 3D distance measurements.

In the static spherically symmetric case the choice of spherical coordinates and static metric dictates the form of three of the gauge fixing coefficients (2): $\bar\Gamma_t = 0$, $\bar\Gamma_\theta = -\cot\theta$, $\bar\Gamma_\phi = 0$, but the form of the quantity $\bar\Gamma_r = \left(\ln\left(\frac{\sqrt{-g_{rr}}}{\sqrt{g_{tt}}\,\rho^2}\right)\right)'$, and, equivalently, the function $\rho(r)$ are still not fixed. Here and further on, the prime denotes differentiation with respect to the variable r. We refer to the freedom of choice of the function $\rho(r)$ as "a *rho-gauge freedom*", and to the choice of the $\rho(r)$ function as "a *rho-gauge fixing*" in a *broad* sense.

In the present article we will not use more general gauge transformations in a broad sense, than the rho-gauge ones. In our 1D approach to the problem at hand all possible mathematical complications, due to the use of such wide class of transformations, can be easily controlled.

The overall action for the aggregate of a point particle and its gravitational field is $\mathcal{A}_{tot} = \mathcal{A}_{GR} + \mathcal{A}_{M_0}$. Neglecting the surface terms one can represent the Hilbert-Einstein action $\mathcal{A}_{GR} = -\frac{1}{16\pi G_n}\int d^4x \sqrt{|g|}R$ and the mechanical action $\mathcal{A}_{M_0} = -M_0\int ds$ of the point source with a bare mass M_0 as integrals with respect to the time t and the radial variable r of the following Lagrangian densities:

$$\mathcal{L}_{GR} = \frac{1}{2G_N}\left(\frac{2\rho\rho'\left(\sqrt{g_{tt}}\right)' + (\rho')^2\sqrt{g_{tt}}}{\sqrt{-g_{rr}}} + \sqrt{g_{tt}}\sqrt{-g_{rr}}\right),$$
$$\mathcal{L}_{M_0} = -M_0\sqrt{g_{tt}}\delta(r). \quad (4)$$

Here G_N is the Newton gravitational constant, $\delta(r)$ is the 1D Dirac function [8]. (We are using units $c=1$.)

As a result of the rho-gauge freedom the field variable $\sqrt{-g_{rr}}$ is not a true dynamical variable but rather plays the role of a Lagrange multiplier, which is needed in a description of a constrained dynamics. This auxiliary variable enters the Lagrangian $\mathcal{L}_{GR}$ in *a nonlinear* manner, in a contrast to the case of the standard Lagrange multipliers. The corresponding Euler-Lagrange equations read:

$$\left(\frac{2\rho\rho'}{\sqrt{-g_{rr}}}\right)' - \frac{\rho'^2}{\sqrt{-g_{rr}}} - \sqrt{-g_{rr}} + 2G_N M_0\delta(r) = 0,$$

$$\left(\frac{(\rho\sqrt{g_{tt}})'}{\sqrt{-g_{rr}}}\right)' - \frac{\rho'\left(\sqrt{g_{tt}}\right)'}{\sqrt{-g_{rr}}} = 0,$$

$$\frac{2\rho\rho'\left(\sqrt{g_{tt}}\right)' + (\rho')^2\sqrt{g_{tt}}}{\sqrt{-g_{rr}}} - \sqrt{g_{tt}}\sqrt{-g_{rr}} \overset{w}{=} 0 \quad (5)$$

where the symbol "$\overset{w}{=}$" denotes a weak equality in the sense of the theory of constrained dynamical systems.

If one ignores the point source of the gravitational field, thus considering only the domain $r > 0$ where $\delta(r) \equiv 0$, one obtains the standard solution of this system [9]:

$$g_{tt}(r) = 1 - \rho_G/\rho(r), \ \ g_{tt}(r)\, g_{rr}(r) = -(\rho(r)')^2, \quad (6)$$

where $\rho_G = 2G_N M$ is the Schwarzschild radius, M is the gravitational (Keplerian) mass of the source, and $\rho(r)$ is an arbitrary $\mathcal{C}^1$ function.

III. SOME EXAMPLES OF DIFFERENT RADIAL GAUGES

In the literature one can find different choices of the function $\rho(r)$ for the problem at hand:

1. Schwarzschild gauge [5]: $\rho(r) = \left(r^3 + \rho_G^3\right)^{1/3}$. It produces $\bar\Gamma_r = -\frac{2}{r}\left(\rho(r^3 - \rho_G^3) - \rho_G(r^3/2 - \rho_G^3)\right)/\left(\rho^3(\rho - \rho_G)\right)$.

2. Hilbert gauge [10]: $\rho(r) = r$. It gives $\bar\Gamma_r = -\frac{2}{r}\left(1 - \frac{\rho_G}{2r}\right)/\left(1 - \frac{\rho_G}{r}\right)$. This simple choice of the function $\rho(r)$ is often related incorrectly with the original Schwarzschild article [5] (See [11]). In this case the coordinate r coincides with the luminosity distance ρ and the physical domain $r \in [0, \infty)$ contains an event horizon at $\rho_H = \rho_G$. This unusual circumstance forces one to develop a nontrivial theory of black holes for Hilbert gauge – see for example [1, 12] and the references therein.

3. Droste gauge [13]: The function $\rho(r)$ is given implicitly by the relations $\rho/\rho_G = \cosh^2\psi \geq 1$ and $r/\rho_G = \psi + \sinh\psi\cosh\psi$. The coefficient $\bar\Gamma_r = -\frac{2}{\rho}\left(\frac{\rho}{\rho_G} - \frac{3}{4}\frac{\rho_G}{\rho}\right)/\sqrt{1 - \frac{\rho_G}{\rho}}$. For this solution the variable r has a clear geometrical meaning: it measures the 3D-radial distance to the center of the spherical symmetry.

4. Weyl gauge [14]: $\rho(r) = \frac{1}{4}\left(\sqrt{r/\rho_G} + \sqrt{\rho_G/r}\right)^2 \geq \rho_G$ and $\bar\Gamma_r = -\frac{2}{r}/\left(1 - \frac{\rho_G^2}{r^2}\right) = -\frac{2}{\rho_G}/\left(\frac{r}{\rho_G} - \frac{\rho_G}{r}\right)$. In this gauge the 3D-space $\mathcal{M}^{(3)}\{-g_{mn}(\mathbf{r})\}$ becomes obviously conformally flat. The coordinate r is the radial variable in the corresponding Euclidean 3D space.

5. Einstein-Rosen gauge [15]. In the original article [15] the variable $u^2 = \rho - \rho_G$ has been used. To have a proper dimension we replace it by $r = u^2 \geq 0$. Hence, $\rho(r) = r + \rho_G \geq \rho_G$ and $\bar{\Gamma}_r = -\frac{2}{r}\frac{r+\rho_G/2}{r+\rho_G}$.

6. Isotropic t-r gauge, defined according to the formula $r = \rho + \rho_G \ln\left(\frac{\rho}{\rho_G} - 1\right)$, $\rho \geq \rho_G$. Then only the combination $(dt^2 - dr^2)$ appears in the 4D-interval ds^2 and $\bar{\Gamma}_r = -\frac{2}{\rho}\left(1 - \frac{\rho_G}{\rho}\right)$.

7. One more rho-gauge in GR was introduced by Pugachev and Gun'ko [9] and, independently, by Menzel [16]. In comparison with the previous ones it is simple and more natural with respect to the quantity $\bar{\Gamma}_r$. We fix it using the condition $\bar{\Gamma}_r = -\frac{2}{r}$ which is identical with the rho-gauge fixing for spherical coordinates in a flat space-time. Thus the curved space-time coordinates t, r, θ, ϕ are fixed in a complete *coherent* way with the flat space-time spherical ones.

Then, absorbing in the coordinates' units two inessential constants, we obtain a novel form of the 4D interval for a point source:

$$ds^2 = e^{2\varphi_N(\mathbf{r})}\left(dt^2 - \frac{dr^2}{N(r)^4}\right) - \frac{r^2}{N(r)^2}\left(d\theta^2 + sin^2\theta d\phi^2\right),$$

where $N(r) = (2\varphi_N)^{-1}\left(e^{2\varphi_N} - 1\right)$; $N(r) \sim 1 + \mathcal{O}(\frac{\rho_G}{r})$ – for $r \to \infty$ and $N(r) \sim \frac{r}{\rho_G}$ – for $r \to +0$. In this specific gauge $\rho(r) = r/N(r)$ and $\rho(r) \sim r$ – for $r \to \infty$, $\rho(r) \to \rho_G$ – for $r \to +0$.

The most remarkable property of this flat space-time-coherent gauge is the role of the *exact* classical Newton gravitational potential $\varphi_N(\mathbf{r}) = -\frac{G_N M}{r}$ in the above *exact* GR solution. As usual, the component $g_{tt} = e^{2\varphi_N(\mathbf{r})} \sim 1 + 2\varphi_N(\mathbf{r}) + \mathcal{O}(\varphi_N(\mathbf{r}))$ – for $r \to \infty$. But now g_{tt} has an essentially singular point at $r = 0$.

The new form of the metric is asymptotically flat for $r \to \infty$. For $r \to +0$ it has a limit $d\theta^2 + sin^2\theta d\phi^2$. The geometry is regular in the whole space-time $\mathcal{M}^{(1,3)}(g_{\mu\nu}(x))$ because the two nonzero scalars $\frac{1}{48}R_{\mu\nu\lambda\kappa}R^{\mu\nu\lambda\kappa} = \frac{1}{4}\left(1 - g_{00}\right)^4 = \frac{1}{4}\left(1 - e^{2\varphi_N(\mathbf{r})}\right)^4$ and $\frac{1}{96}R_{\mu\nu\lambda\kappa}R^{\lambda\kappa}{}_{\sigma\tau}R^{\sigma\tau\mu\nu} = -\frac{1}{8}\left(1 - g_{00}\right)^6 = -\frac{1}{8}\left(1 - e^{2\varphi_N(\mathbf{r})}\right)^6$ are finite everywhere, including the center $r = +0$. At this center we have a zero 4D volume, because $\sqrt{|g|} = e^{4\varphi_N(\mathbf{r})}N(r)^{-8}r^2\sin\theta$. At the same time the coordinates $x^m \in (-\infty, \infty)$, $m = 1, 2, 3$; are globally defined in $\mathcal{M}^{(1,3)}(g_{\mu\nu}(x))$. One can check that $\Delta_g x^m = 2\left(\frac{N}{r}\right)^3 x^m$. Hence, $\Delta_g x^m \sim \mathcal{O}\left(\frac{1}{r^2}\right)$ – for $r \to \infty$ and $\Delta_g x^m \sim 2x^m \sim \mathcal{O}(r)$ – for $r \to +0$, i.e., the coherent coordinates x^m are asymptotically harmonic in both limits, but not for finite values of $r \approx \rho_G$.

At this point one has to analyze two apparent facts:

i) An event horizon ρ_H exists in the physical domain *only* under Hilbert choice of the function $\rho(r) \equiv r$, but not in the other gauges, discussed above. This demonstrates that the existence of black holes strongly depends on this choice of the rho-gauge in a *broad* sense.

ii) The choice of the function $\rho(r)$ can change drastically the character of the singularity at the place of the point source of the metric field in GR.

New interesting gauge conditions for the radial variable r and corresponding gauge transformations were introduced and investigated in [17]. In the recent articles [18] it was shown that using a nonstandard ρ-gauge for the applications of GR to the stelar physics one can describe extreme objects with *arbitrary large* mass, density and size. In particular, one is able to shift the value of the Openheimer-Volkoff limiting mass $0.7M_\odot$ of neutron stars to a new one: $3.8M_\odot$. This may shed a new light on these astrophysical problems and once more shows that the choice of rho-gauge (i.e. the choice of the coordinate r) can have real physical consequences.

Thus we see that by analogy with the classical electrodynamics and non-abelian gauge theories of general type, in GR we must use indeed a more refined terminology and corresponding mathematical constructions. We already call the choice of the function $\rho(r)$ "a rho-gauge fixing in a broad sense". Now we see that the different functions $\rho(r)$ may describe *different* physical solutions of Einstein equations (1) with the *same* spherical symmetry in presence of only one singular point at the center of symmetry. As we have seen, the mathematical properties of the singular point may depend on the choice of the rho-gauge in a broad sense. Now the problem is to clarify the physical meaning of the singular points with different mathematical characteristics.

Some of the static spherically symmetric solutions with a singularity at the center can be related by regular gauge transformations. For example, choosing the Hilbert gauge we obtain a black hole solution of the *vacuum* Einstein equations (1). Then we can perform a *regular* gauge transformation to harmonic coordinates, defined by the relations $\bar{\Gamma}_t = 0$, $\bar{\Gamma}_{x^1} = 0$, $\bar{\Gamma}_{x^2} = 0$, $\bar{\Gamma}_{x^3} = 0$, preserving the existence of the event horizon, i.e. staying in the same gauge sector. In the next sections we will find a geometrical criteria for answering the question: "When the different representations of static spherically symmetric solutions with point singularity describe diffeomorphic space-times ?"

The definition of the regular gauge transformations allows transformations that do not change the number and the character of singular points of the solution in the real domain and the behavior of the solution at the boundary, but can place these points and the boundary at new positions.

IV. GENERAL FORM OF THE FIELD EQUATIONS IN HILBERT GAUGE AND SOME BASIC PROPERTIES OF THEIR SOLUTIONS

According to Lichnerowicz [2, 20, 21], the physical space-times $\mathcal{M}^{(1,3)}\{g_{\mu\nu}(x)\}$ of general type must be smooth manifolds at least of class C^2 and the metric coefficients $g_{\mu\nu}(x)$ have to be at least of class C^3, i.e. at

least three times continuously differentiable functions of the coordinates x. When one considers $g_{tt}(r)$, $g_{rr}(r)$ and $\rho(r)$ not as distributions, but as usual functions of this class, one is allowed to use the rules of the analysis of functions of one variable r. Especially, one can multiply these functions and their derivatives of the corresponding order, one can raise them at various powers, define functions like log, exp and other mathematical functions of $g_{tt}(r)$, $g_{rr}(r)$, $\rho(r)$ and coresponding derivatives. In general, these operations are forbidden for distributions [8].

In addition, making use of the standard rules for operating with Dirac δ-function [8] and accepting (for simplicity) the following assumptions: 1) the function $\rho(r)$ is a monotonic function of the *real* variable r; 2) $\rho(0) = \rho_0$ and the equation $\rho(r) = \rho_0$ has only one *real* solution: $r = 0$; one obtains the following alternative form of the Eq.(5):

$$\left(\sqrt{-g_{\rho\rho}} - \frac{1}{\sqrt{-g_{\rho\rho}}}\right)\frac{d}{d\ln\rho}\ln\left(\rho\left(\frac{1}{-g_{\rho\rho}} - 1\right)\right) =$$
$$2\sigma_0 G_N M_0\,\delta(\rho - \rho_0),$$

$$\frac{d^2\ln g_{tt}}{(d\ln\rho)^2} + \frac{1}{2}\left(\frac{d\ln g_{tt}}{d\ln\rho}\right)^2 + \left(1 + \frac{1}{2}\frac{d\ln g_{tt}}{d\ln\rho}\right)\frac{d\ln g_{\rho\rho}}{d\ln\rho} = 0,$$

$$\frac{d\ln g_{tt}}{d\ln\rho} + g_{\rho\rho} + 1 \overset{w}{=} 0. \quad (7)$$

Note that the only remnant of the function $\rho(r)$ in the system Eq.(7) are the numbers ρ_0 and $\sigma_0 = sign(\rho'(0))$, which enter only the first equation, related to the source of gravity. Here g_{tt} and $g_{\rho\rho} = g_{rr}/(\rho')^2$ are considered as functions of the independent variable $\rho \in [\rho_0, \infty)$.

The form of the Eq. (7) shows why one is tempted to consider the Hilbert gauge as a preferable one: In this form the arbitrary function $\rho(r)$ "disappears".

One usually ignores the general case of an arbitrary value $\rho_0 \neq 0$ accepting, without any justification, the value $\rho_0 = 0$, which seems to be natural in Hilbert gauge. Indeed, if we consider the luminosity distance as a measure of the *real* geometrical distance to the point source of gravity in the 3D space, we have to accept the value $\rho_0 = 0$ for the position of the point source. Otherwise the δ-function term in the Eq. (7) will describe a *shell* with radius $\rho_0 \neq 0$, instead of a *point* source.

Actually, according to our previous consideration the point source has to be described using the function $\delta(r)$. The physical source of gravity is placed at the point $r = 0$ by definition. There is no reason to change this initial position of the source, or the interpretation of the variables in the problem at hand. To what value of the luminosity distance $\rho_0 = \rho(0)$ corresponds the *real* position of the point source is not known *a priori*. This depends strongly on the choice of the rho-gauge. One can not exclude such a nonstandard behavior of the *physically reasonable* rho-gauge function $\rho(r)$ which leads to some value $\rho_0 \neq 0$. Physically this means that instead to infinity, the luminosity of the *point* source will go to a *finite* value, when

the distance to the source goes to zero. This very interesting new possibility appears in curved space-times due to their unusual geometrical properties. It may have a great impact for physics and deserves further careful investigation.

In contrast, if one accepts the value $\rho_0 = 0$, one has to note that the Hilbert-gauge singularity at $\rho = 0$ will be *space-like,* not time-like. This is a quite unusual *nonphysical* property for a physical source of any static physical field.

The choice of the values of the parameters ρ_0 and σ_0 was discussed in [22]. There, a new argument in favor of the choice $\rho_0 = \rho_G$ and $\sigma_0 = 1$ was given for different physical problems with spherical symmetry using an analogy with the Newton theory of gravity. In contrast, in the present article we shall analyze the physical meaning of the different values of the parameter $\rho_0 \geq 0$. They turn out to be related with the gravitational mass defect.

The solution of the subsystem formed by the last two equations of the system (7) is given by the well known functions

$$g_{tt}(\rho) = 1 - \rho_G/\rho, \quad g_{\rho\rho}(\rho) = -1/g_{tt}(\rho). \quad (8)$$

Note that in this subsystem one of the equations is a field equation, but the other one is a constraint. However, these functions do not solve the first of the Eq.(7) for any value of ρ_0, if $M_0 \neq 0$. Indeed, for these functions the left hand side of the first field equation equals identically zero and does not have a δ-function-type of singularity, in contrast to the right hand side. Hence, the first field equation remains unsolved. Thus we see that:

1) Outside the singular point, i.e. for $\rho > \rho_0$, the Birkhoff theorem is strictly valid and we have a standard form of the solution, when expressed through the luminosity distance ρ.

2) The assumption that $g_{tt}(r)$, $g_{rr}(r)$ and $\rho(r)$ are usual C^3 smooth functions, instead of distributions, yields a contradiction, if $M_0 \neq 0$. This way one is not able to describe correctly the gravitational field of a massive point source of gravity in GR. For this purpose the first derivative with respect to the variable r at least of one of the metric coefficients $g_{\mu\nu}$ must have a finite jump, needed to reproduce the Dirac δ-function in the energy-momentum tensor of the *massive* point particle.

3) The widespread form of the Schwarzschild's solution in Hilbert gauge (8) does not describe a gravitational field of a massive point source and corresponds to the case $M_0 = 0$. It does not belong to any physical gauge sector of gravitational field, created by a massive point source in GR. Obviously, this solution has a geometrical-topological nature and may be used in the attempts to reach pure geometrical description of "matter without matter". In the standard approach to this solution no bare mass distribution was ever introduced.

The above consideration confirms the conclusion, which was reached in [19] using isotropic gauge for Hilbert solution. In contrast to [19], in the next Sections we will show how one can include in GR *neutral* particles

with nonzero bare mass M_0. The corresponding new solutions are related to the Hilbert one via singular gauge transformations.

V. NORMAL COORDINATES FOR GRAVITATIONAL FIELD OF A MASSIVE POINT PARTICLE IN GENERAL RELATIVITY

An obstacle for the description of the gravitational field of a point source at the initial stage of development of GR was the absence of an adequate mathematical formalism. Even after the development of the correct theory of mathematical distributions [8] there still exist an opinion that this theory is inapplicable to GR because of the nonlinear character of Einstein equations (1). For example, the author of the article [21] emphasizes that "the Einstein equations, being non-linear, are defined essentially, only within framework of functions. The functionals, introduced in ... physics and mathematics (Dirac's δ-function, "weak" solutions of partial differential equations, distributions of Scwartz) are suitable only for linear problems, since their product is not, in general, defined." In the more recent article [23] the authors have considered singular lines and surfaces, using mathematical distributions. They have stressed, that "there is apparently no viable treatment of point particles as concentrated sources in GR". See also [2] and the references therein. Here we propose a novel approach to this problem, based on a specific choice of the field variables.

Let us represent the metric ds^2 (3) of the problem at hand in a specific form:

$$e^{2\varphi_1}dt^2 - e^{-2\varphi_1+4\varphi_2-2\bar{\varphi}}dr^2 - \bar{\rho}^2 e^{-2\varphi_1+2\varphi_2}(d\theta^2 + \sin^2\theta d\phi^2)\,(9)$$

where $\varphi_1(r)$, $\varphi_2(r)$ and $\bar{\varphi}(r)$ are unknown functions of the variable r and $\bar{\rho}$ is a constant – the unit for luminosity distance $\rho = \bar{\rho}\,e^{-\varphi_1+\varphi_2}$. By ignoring the surface terms in the corresponding integrals, one obtains the gravitational and the mechanical actions in the form:

$$\mathcal{A}_{GR} = \frac{1}{2G_N}\int dt \int dr \left(e^{\bar{\varphi}}\left(-(\bar{\rho}\varphi_1')^2 + (\bar{\rho}\varphi_2')^2\right) + e^{-\bar{\varphi}}e^{2\varphi_2}\right),$$

$$\mathcal{A}_{M_0} = -\int dt \int dr\, M_0\, e^{\varphi_1}\delta(r). \tag{10}$$

Thus we see that the field variables $\varphi_1(r)$, $\varphi_2(r)$ and $\bar{\varphi}(r)$ play the role of *a normal fields' coordinates* in our problem. The field equations read:

$$\bar{\Delta}_r\varphi_1(r) = \frac{G_N M_0}{\bar{\rho}^2}e^{\varphi_1(r)-\bar{\varphi}(r)}\delta(r),$$

$$\bar{\Delta}_r\varphi_2(r) = \frac{1}{\bar{\rho}^2}e^{2(\varphi_2(r)-\bar{\varphi}(r))} \tag{11}$$

where $\bar{\Delta}_r = e^{-\bar{\varphi}}\frac{d}{dr}\left(e^{\bar{\varphi}}\frac{d}{dr}\right)$ is related to the radial part of the 3D-Laplacean, $\Delta = -1/\sqrt{|^{(3)}g|}\partial_i\left(\sqrt{|^{(3)}g|}g^{ij}\partial_j\right) = -g^{ij}\left(\partial_{ij}^2 - \bar{\Gamma}_i\partial_j\right) = \Delta_r + 1/\rho^2\Delta_{\theta\phi}:\ \bar{\Delta}_r = -g_{rr}\Delta_r.$

The variation of the total action with respect to the auxiliary variable $\bar{\varphi}$ gives the constraint:

$$e^{\bar{\varphi}}\left(-(\bar{\rho}\varphi_1')^2 + (\bar{\rho}\varphi_2')^2\right) - e^{-\bar{\varphi}}e^{2\varphi_2} \overset{w}{=} 0. \tag{12}$$

One can have some doubts about the correctness of the above derivation of the field equations (11), because here we use the Weyl's trick, applying the spherical symmetry directly to the action functional, not to the Einstein equations (1). The correctness of the result of this procedure is proved in the Appendix A. Therefore, if one prefers, one can consider the Lagrangian densities (4), or the actions (10) as an auxiliary tools for formulating of our one-dimensional problem, defined by the reduced spherically symmetric Einstein equations (1), as a variational problem. The variational approach makes transparent the role, in the sense of the theory of constrained dynamical systems, of the various differential equations, which govern the problem.

VI. REGULAR GAUGES AND GENERAL REGULAR SOLUTIONS OF THE PROBLEM

The advantage of the above normal fields' coordinates is that when expressed through them the field equations (11) are linear with respect to the derivatives of the unknown functions $\varphi_{1,2}(r)$. This circumstance legitimates the correct application of the mathematical theory of distributions and makes our normal coordinates privileged field variables.

The choice of the function $\bar{\varphi}(r)$ fixes the rho-gauge in the normal coordinates. We have to choose this function in a way that makes the first of the equations (11) meaningful. Note that this non-homogeneous equation is quasi-linear and has a correct mathematical meaning if, and only if, the condition $|\varphi_1(0) - \bar{\varphi}(0)| < \infty$ is satisfied.

Let's consider once more the domain $r > 0$. In this domain the first of the equations (11) gives $\varphi_1(r) = C_1\int e^{-\bar{\varphi}(r)}dr + C_2$ with arbitrary constants $C_{1,2}$. Suppose that the function $\bar{\varphi}(r)$ has an asymptotics $\exp(-\bar{\varphi}(r)) \sim kr^n$ in the limit $r \to +0$ (with some arbitrary constants k and n). Then one easily obtains $\varphi_1(r) - \bar{\varphi}(r) \sim C_1 kr^{n+1}/(n+1) + n\ln r + \ln k + C_2$ – if $n \neq -1$, and $\varphi_1(r) - \bar{\varphi}(r) \sim (C_1 k - 1)\ln r + \ln k + C_2$ – for $n = -1$. Now we see that one can satisfy the condition $\lim_{r\to 0}|\varphi_1(r) - \bar{\varphi}(r)| = constant < \infty$ for arbitrary values of the constants $C_{1,2}$ if, and only if $n = 0$. This means that we must have $\bar{\varphi}(r) \sim k = const \neq \pm\infty$ for $r \to 0$. We call such gauges *regular gauges* for the problem at hand. Then $\varphi_1(0) = const \neq \pm\infty$. Obviously, the simplest choice of a regular gauge is $\bar{\varphi}(r) \equiv 0$. Further on we shall use this *basic regular gauge*. Other regular gauges for the same gauge sector defer from it by a regular rho-gauge transformation which describes a diffeomorphism of the fixed by the basic regular gauge manifold $\mathcal{M}^{(3)}\{g_{mn}(\mathbf{r})\}$. In terms of the metric components the basic regular gauge condition reads $\rho^4 g_{tt} + \bar{\rho}^4 g_{rr} = 0$ and gives $\bar{\Gamma}_r = 0$.

Under this gauge the field equations (11) acquire the simplest quasi-linear form:

$$\varphi_1''(r) = \frac{G_N M_0}{\bar{\rho}^2} e^{\varphi_1(0)} \delta(r), \quad \varphi_2''(r) = \frac{1}{\bar{\rho}^2} e^{2\varphi_2(r)}. \quad (13)$$

The constraint (12) acquires the simple form:

$$-(\bar{\rho}\varphi_1')^2 + (\bar{\rho}\varphi_2')^2 - e^{2\varphi_2} \stackrel{w}{=} 0. \quad (14)$$

As can be easily seen, the basic regular gauge $\bar{\varphi}(r) \equiv 0$ has the unique property to split the system of field equations (11) and the constraint (12) into three independent relations.

The new field equations (13) have a general solution

$$\varphi_1(r) = \frac{G_N M_0}{\bar{\rho}^2} e^{\varphi_1(0)} \big(\Theta(r) - \Theta(0)\big) r + \varphi_1'(0) r + \varphi_1(0),$$

$$\varphi_2(r) = -\ln\left(\frac{1}{\sqrt{2\varepsilon_2}} \sinh\left(\sqrt{2\varepsilon_2}\frac{r_\infty - r}{\bar{\rho}}\right)\right). \quad (15)$$

The first expression in Eq.(15) represents a distribution $\varphi_1(r)$. In it $\Theta(r)$ is the Heaviside step function. Here we use the *additional assumption* $\Theta(0) := 1$. It gives a specific regularization of the products, degrees and functions of the distribution $\Theta(r)$ and makes them definite. For example: $(\Theta(r))^2 = \Theta(r)$, $(\Theta(r))^3 = \Theta(r)$, ..., $\Theta(r)\delta(r) = \delta(r)$, $f(r\Theta(r)) = f(r)\Theta(r) - f(0)\big(\Theta(r) - \Theta(0)\big)$ – for any function $f(r)$ with a convergent Taylor series expansion around the point $r = 0$, and so on. This is the only simple regularization of distributions we need in the present article.

The second expression $\varphi_2(r)$ in Eq.(15) is a usual function of the variable r. The symbol r_∞ is used as an abbreviation for the constant expression $r_\infty = sign\left(\varphi_2'(0)\right) \bar{\rho} \sinh\left(\sqrt{2\varepsilon_2}e^{-\varphi_2(0)}\right)/\sqrt{2\varepsilon_2}$.

The constants

$$\varepsilon_1 = -\frac{1}{2}\bar{\rho}^2\varphi_1'(r)^2 + \frac{G_N M_0}{\bar{\rho}^2}\varphi_1'(0) e^{\varphi_1(0)}\big(\Theta(r) - \Theta(0)\big),$$

$$\varepsilon_2 = \frac{1}{2}\left(\bar{\rho}^2\varphi_2'(r)^2 - e^{2\varphi_2(r)}\right) \quad (16)$$

are the values of the corresponding first integrals (16) of the differential equations (13) for a given solution (15).

Then for the regular solutions (15) the condition(14) reads:

$$\varepsilon_1 + \varepsilon_2 + \frac{G_N M_0}{\bar{\rho}^2}e^{\varphi_1(0)}\big(\Theta(r) - \Theta(0)\big) \stackrel{w}{=} 0. \quad (17)$$

An unexpected property of this relation is that it cannot be satisfied for any value of the variable $r \in (-\infty, \infty)$, because $\varepsilon_{1,2}$ are constants. The constraint (17) can be satisfied either on the interval $r \in [0, \infty)$, or on the interval $r \in (-\infty, 0)$. If, from physical reasons we chose it to be valid at only one point $r^* \in [0, \infty)$, this relation will be satisfied on the whole interval $r \in [0, \infty)$ and this interval will be the physically admissible real domain of the radial variable. Thus one can see that our approach

gives a unique possibility *to derive* the admissible real domain of the variable r from the dynamical constraint (14), i.e., this dynamical constraint yields a geometrical constraint on the values of the radial variable.

As a result, in the physical domain the values of the first integrals (16) are related by the standard equation

$$\varepsilon_{tot} = \varepsilon_1 + \varepsilon_2 \stackrel{w}{=} 0, \quad (18)$$

which reflects the fact that our variation problem is invariant under re-parametrization of the independent variable r. At the end, as a direct consequence of the relation (18) one obtains the inequality $\varepsilon_2 = -\varepsilon_1 > 0$, because in the real physical domain $r \in [0, \infty)$ we have $\varepsilon_1 = -\frac{1}{2}\bar{\rho}^2\varphi_1'(r)^2 = const < 0$.

For the function $\rho_{reg,0}(r) \geq 0$, which corresponds to the basic regular gauge, one easily obtains

$$\rho_{reg|_{\bar{\varphi}=0}}(r) = \rho_G \left(1 - \exp\left(4\frac{r - r_\infty}{\rho_G}\right)\right)^{-1}. \quad (19)$$

Now one has to impose several additional conditions on the solutions (15):

i) The requirement to have an asymptotically flat space-time. The limit $r \to r_\infty$ corresponds to the limit $\rho \to \infty$. For solutions (15) we have the property $g_{rr}(r_\infty)/\rho'^2(r_\infty) = 1$. The only nontrivial asymptotic condition is $g_{tt}(r_\infty) = 1$. It gives $\varphi_1'(0) r_\infty + \varphi_1(0) = 0$.

ii) The requirement to have the correct Keplerian mass M, as seen by a distant observer. Excluding the variable $r > 0$ from $g_{tt}(r) = e^{2\varphi_1(r)}$ and $\rho(r) = \bar{\rho}e^{-\varphi_1(r) + \varphi_2(r)}$ for solutions (15) one obtains $g_{tt} = 1 + \frac{const}{\rho}$, where $const = 2\, sign(\rho)\, sign(\varphi_2'(0))\, \bar{\rho}^2\, \varphi_1'(0) = -2G_N M$.

iii) The consistency of the previous conditions with the relation $g_{tt}\, g_{rr} + \rho'^2 = 0$ gives $sign(r_\infty) = sign(\rho) = sign(\varphi_1'(0)) = sign(\varphi_2'(0)) = 1$.

iv) The most suitable choice of the unit $\bar{\rho}$ is $\bar{\rho} = G_N M = \rho_G/2$. As a result all initial constants become functions of the two parameters in the problem – r_∞ and M: $\varphi_1(0) = -\frac{r_\infty}{G_N M}$, $\varphi_2(0) = -\ln\left(\sinh\left(\frac{r_\infty}{G_N M}\right)\right)$, $\varphi_1'(0) = -\frac{1}{G_N M}$, $\varphi_2'(0) = \frac{1}{G_N M}\coth\left(\frac{r_\infty}{G_N M}\right)$.

v) The gravitational defect of the mass of a point particle.

Representing the bare mechanical mass M_0 of the point source in the form $M_0 = \int_0^{r_\infty} M_0\,\delta(r)dr = 4\pi\int_0^{r_\infty}\sqrt{-g_{rr}(r)}\,\rho^2(r)\,\mu(r)dr$, one obtains for the mass distribution of the point particle the expression $\mu(r) = M_0\,\delta(r)/\left(4\pi\sqrt{-g_{rr}(r)}\,\rho^2(r)\right) = M_0\,\delta_g(r)$, where $\delta_g(r) := \delta(r)/\left(4\pi\sqrt{-g_{rr}(r)}\rho^2(r)\right)$ is the 1D *invariant* Dirac delta function. The Keplerian gravitational mass M can be calculated using the Tolman formula [1]:

$$M = 4\pi\int_0^{r_\infty}\rho'(r)\rho^2(r)\mu(r)dr = M_0\sqrt{g_{tt}(0)}. \quad (20)$$

Here we use the relation $\rho' = \sqrt{-g_{tt}\,g_{rr}}$. As a result we reach the relations: $g_{tt}(0) = e^{2\varphi_1(0)} = \exp\left(-2\frac{r_\infty}{G_N M}\right) =$

$\left(\frac{M}{M_0}\right)^2 \leq 1$ and $r_\infty = G_N M \ln\left(\frac{M_0}{M}\right) \geq 0$. (Note that due to our convention $\Theta(0) := 1$ the component $g_{tt}(r)$ is a continuous function in the interval $r \in [0,\infty)$ and $g_{tt}(0) = g_{tt}(+0)$ is a well defined quantity.)

The ratio $\varrho = \frac{M}{M_0} = \sqrt{g_{tt}(0)} \in [0,1]$ describes the gravitational mass defect of the point particle as a second physical parameter in the problem. The Keplerian mass M and the ratio ϱ define completely the solutions (15).

For the initial constants of the problem one obtains:

$$\varphi_1(0) = \ln \varrho, \quad \varphi_2(0) = -\ln\frac{1-\varrho^2}{2\varrho},$$

$$\varphi_1'(0) = \frac{1}{G_N M}, \quad \varphi_2'(0) = \frac{1}{G_N M}\frac{1+\varrho^2}{1-\varrho^2}. \qquad (21)$$

Thus we arrive at the following form of the solutions (15):

$$\varphi_1(r) = \frac{r\,\Theta(r)}{G_N M} - \ln(1/\varrho),$$

$$\varphi_2(r) = -\ln\left(\frac{1}{2}\left(1/\varrho\, e^{-r/G_N M} - \varrho\, e^{r/G_N M}\right)\right) \qquad (22)$$

and the rho-gauge fixing function

$$\rho_{reg|\bar\varphi=0}(r) = \rho_G \left(1 - \varrho^2 \exp\left(4r/\rho_G\right)\right)^{-1}. \qquad (23)$$

An unexpected feature of this *two parametric* variety of solutions for the gravitational field of a point particle is that each solution must be considered only in the domain $r \in [0,\,G_N M \ln(1/\varrho)]$, if we wish to have a monotonic increase of the luminosity distance in the interval $[\rho_0, \infty)$.

It is easy to check that away from the source, i.e., for $r > 0$, these solutions coincide with the solution (8) in the Hilbert gauge. Hence, outside the source the solutions (22) acquire the well known standard form, when represented using the variable ρ. This means that the solutions (22) strictly respect a generalized Birkhoff theorem. Its proper generalization requires only a justification of the physical domain of the variable ρ. It is remarkable that for the solutions (22) the minimal value of the luminosity distance is

$$\rho_0 = 2G_N M/(1 - \varrho^2) \geq \rho_G. \qquad (24)$$

This changes the Gauss theorem and leads to important physical consequences. One of them is that one must apply the Birkhoff theorem only in the interval $\rho \in [\rho_0, \infty)$.

VII. REGULAR GAUGE MAPPING OF THE INTERVAL $r \in [0, r_\infty]$ ONTO THE WHOLE INTERVAL $r \in [0, \infty]$

As we have stressed in the previous section, the solutions (22) for the gravitational field of a point particle must be considered only in the physical domain

$r \in [0, r_\infty]$. It does not seem to be very convenient to work with such unusual radial variable r. One can easily overcome this problem using the regular rho-gauge transformation

$$r \to r_\infty \frac{r/\tilde r}{r/\tilde r + 1} \qquad (25)$$

with an arbitrary scale $\tilde r$ of the new radial variable r (Note that in the present article we are using the same notation r for different radial variables.) This linear fractional diffeomorfism does not change the number and the character of the singular points of the solutions in the whole compactified complex plane $\tilde{\mathcal{C}}_r^{(1)}$ of the variable r. The transformation (25) simply places the point $r = r_\infty$ at the infinity $r = \infty$, at the same time preserving the initial place of the origin $r = 0$. Now the new variable r varies in the standard interval $r \in [0, \infty)$, the regular solutions (22) acquire the final form

$$\varphi_1(r) = -\ln(1/\varrho)\left(1 - \frac{r/\tilde r}{r/\tilde r + 1}\Theta(r/\tilde r)\right),$$

$$\varphi_2(r) = -\ln\left(\frac{1}{2}\left((1/\varrho)^{\frac{1}{r/\tilde r + 1}} - \varrho^{\frac{1}{r/\tilde r + 1}}\right)\right),$$

$$\bar\varphi(r) = 2\ln(r/\tilde r + 1) + \ln(\tilde r/r_\infty). \qquad (26)$$

The final form of the rho-gauge fixing function reads:

$$\rho_{reg}(r) = \rho_G\left(1 - \varrho^{\frac{2}{r/\tilde r + 1}}\right)^{-1}. \qquad (27)$$

The last expression shows that the mathematically admissible interval of the values of the ratio ϱ is the *open* interval $(0,1)$. This is so, because for $\varrho = 0$ and for $\varrho = 1$ we would have impermissible trivial gauge-fixing functions $\rho_{reg}(r) \equiv 1$ and $\rho_{reg}(r) \equiv 0$, respectively.

Now we are ready to describe the singular character of the coordinate transition from the Hilbert form of Schwarzschild solution (8) to the regular one (26). To simplify notations, let us introduce dimensionless variables $z = \rho/\rho_g$ and $\zeta = r/\tilde r$. Then the Eq. (27) shows that the essential part of the change of the coordinates is described, in both directions, by the functions:

$$z(\zeta) = \left(1 - \varrho^{\frac{2}{\zeta+1}}\right)^{-1}, \text{ and}$$

$$\zeta(z) = \frac{\ln(1/\varrho^2)}{\ln z - \ln(z-1)} - 1, \quad \varrho \in (0,1). \qquad (28)$$

Obviously, the function $z(\zeta)$ is regular at the place of the point source $\zeta = 0$; it has a simple pole at $\zeta = \infty$ and an essential singular point at $\zeta = -1$. At the same time the inverse function $\zeta(z)$ has a logarithmic branch points at the Hilbert-gauge center of symmetry $z = 0$ and at the "event horizon" $z = 1$. Thus we see how one produces the Hilbert-gauge singularities at $\rho = 0$ and at $\rho = \rho_H$, starting from a regular solution. The derivative

$$d\zeta/dz = \frac{\ln(1/\varrho^2)}{z(z-1)\left(\ln z - \ln(z-1)\right)^2}$$

aproaches infinity at these two points, hence the singular character of the change in the whole complex domain of the variables. Thus we reach a complete description of the change of coordinates and its singularities in the complex domain of the radial variable.

The restriction of the change of the radial variables on the corresponding *physical* interval outside the source: $\zeta \in (0, \infty) \leftrightarrow z \in (1/(1-\varrho^2), \infty)$, is a regular one.

The expressions (26) and (27) still depend on the choice of the units for the new variable r. We have to fix the arbitrary scale of this variable in the form $\tilde{r} = \rho_G / \ln(1/\varrho^2) = G_N M / \ln\left(\frac{M_0}{M}\right)$ to ensure the validity of the standard asymptotic expansion: $g_{tt} \sim 1 - \rho_G/r + \mathcal{O}\left((\rho_G/r)^2\right)$ when $r \rightarrow \infty$.

Then the final form of the 4D interval, defined by the new regular solutions outside the source is:

$$ds^2 = e^{2\varphi_G}\left(dt^2 - \frac{dr^2}{N_G(r)^4}\right) - \rho(r)^2\left(d\theta^2 + \sin^2\theta d\phi^2\right). \quad (29)$$

Here we are using a modified (Newton-like) gravitational potential:

$$\varphi_G(r; M, M_0) := -\frac{G_N M}{r + G_N M / \ln(\frac{M_0}{M})} \quad (30)$$

a coefficient $N_G(r) = (2\varphi_G)^{-1}\left(e^{2\varphi_G} - 1\right)$, and an optical luminosity distance

$$\rho(r) = 2G_N M / \left(1 - e^{2\varphi_G}\right) = \frac{r + G_N M / \ln(\frac{M_0}{M})}{N_G(r)}. \quad (31)$$

These basic formulas describe in a more usual way our regular solutions of Einstein equations for $r \in (0, \infty)$. They show immediately that in the limit $\varrho \rightarrow 0$ our solutions tend to the Pugachev-Gun'ko-Menzel one, and $\varphi_G(0; M, M_0) = \ln\varrho \rightarrow -\infty$. In the limit: $\varrho \rightarrow 1$ we obtain for any value of the ratio r/ρ_g: $g_{tt}(r/\rho_G, \varrho) \rightarrow 1$, $g_{rr}(r/\rho_G, \varrho) \rightarrow -1$, and $\rho(r/\rho_G, \varrho) \rightarrow \infty$. Because of the last result the 4D geometry does not have a meaningful limit when $\varrho \rightarrow 1$. In this case $\varphi_G(r; M, M_0) \rightarrow 0$ at all 3D space points. Physically this means that solutions without mass defect are not admissible in GR.

VIII.   TOTAL ENERGY OF A POINT SOURCE AND ITS GRAVITATIONAL FIELD

In the problem at hand we have an extreme example of an "island universe". In it a privileged reference system and a well defined global time exist. It is well known that under these conditions the energy of the gravitational field can be defined unambiguously [1]. Moreover, we can calculate the total energy of the aggregate of a mechanical particle and its gravitational field.

Indeed, the canonical procedure produces a total Hamilton density $\mathcal{H}_{tot} = \Sigma_{a=1,2;\mu=t,r}\,\pi_a^\mu\,\varphi_{a,\mu} - \mathcal{L}_{tot} = \frac{1}{2G_N}\left(-\bar{\rho}^2\varphi_1'^2 + \bar{\rho}^2\varphi_2'^2 - e^{2\varphi_2}\right) + M_0 e^{\varphi_1}\delta(r)$. Using the

constraint (18) and the first of the relations (21), one immediately obtains for the total energy of the GR universe with one point particle in it:

$$E_{tot} = \int_0^\infty \mathcal{H}_{tot}dr = M = \varrho M_0 \leq M_0. \quad (32)$$

This result completely agrees with the strong equivalence principle of GR. The energy of the gravitational field, created by a point particle is the negative quantity: $E_{GR} = E_{tot} - E_0 = M - M_0 = -M_0(1 - \varrho) < 0$.

The above consideration gives a clear physical explanation of the gravitational mass defect for a point particle.

IX.   INVARIANT CHARACTERISTICS OF THE SOLUTIONS WITH A POINT SOURCE

A.   Local Singularities of Point Sources

Using the invariant 1D Dirac function one can write down the first of the Eq. (11) in a form of an *exact relativistic* Poisson equation:

$$\Delta_r\varphi_1(r) = 4\pi G_N M_0\,\delta_g(r). \quad (33)$$

This equation is a specific realization of Fock's idea (see Fock in [1]) using our normal fields coordinates (Sec. V). It can be re-written, too, in a transparent 3D form:

$$\Delta\varphi_1(r) = 4\pi G_N M_0\,\delta_g^{(3)}(\mathbf{r}). \quad (34)$$

The use of the invariant Dirac δ_g-function has the advantage that under diffeomorphisms of the space $\mathcal{M}^{(3)}\{-g_{mn}(\mathbf{r})\}$ the singularities of the right hand side of the relativistic Poisson equation (33) remain unchanged. Hence, we have the same singularity at the place of the source for the whole class of physically equivalent rho-gauges. Then one can distinguish the physically different solutions of Einstein equations (1) with spherical symmetry by investigating the asymptotics in the limit $r \rightarrow +0$ of the coefficient $\gamma(r) = 1/\left(4\pi\rho(r)^2\sqrt{-g_{rr}(r)}\right)$ in front of the usual 1D Dirac $\delta(r)$-function in the representation $\delta_g(r) = \gamma(r)\delta(r)$ of the invariant one.

For regular solutions (22), (26) the limit $r \rightarrow +0$ of this coefficient is a constant

$$\gamma(0) = \frac{1}{4\pi\rho_G^2}\left(\frac{1}{\varrho} - \varrho\right), \quad (35)$$

which describes the intensity of the invariant δ-function and leads to the formula $\delta_g(r) = \gamma(0)\delta(r)$. We see that:

1) The condition $\varrho \in (0, 1)$ ensures the correct sign of the intensity $\gamma(0)$, i.e., the property $\gamma(0) \in (0, \infty)$, and thus a negativity of the total energy E_{GR} of the gravitational field of a point particle, according the previous Section III.

2) For $\varrho = 0$ we have $\gamma(0) = \infty$, and for $\varrho = 1$: $\gamma(0) = 0$. Hence, we have non-physical values of the intensity $\gamma(0)$

in these two cases. The conclusion is that *the physical interval* of values of the ratio ϱ is the open interval $(0,1)$. This is consistent with the mathematical analysis of this problem, given in the previous Section.

It is easy to obtain the asymptotics of the coefficient $\gamma(r)$ in the limit $r \to 0$ for other solutions, considered in the Introduction:

Schwarzschild solution: $\quad \gamma_S(r) \sim \frac{1}{4\pi\rho_G^2}\left(\frac{\rho_G}{r}\right)^{1/2}$;

Hilbert solution: $\quad \gamma_H(r) \sim \frac{i}{4\pi\rho_G^2}\left(\frac{\rho_G}{r}\right)^{5/2}$;

Droste solution: $\quad \gamma_D(r) \sim \frac{1}{4\pi\rho_G^2}$;

Weyl solution: $\quad \gamma_W(r) \sim \frac{16}{\pi\rho_G^2}\left(\frac{r}{\rho_G}\right)^4$;

Einstein-Rosen solution: $\quad \gamma_{ER}(r) \sim \frac{1}{4\pi\rho_G^2}\left(\frac{\rho_G}{r}\right)^{1/2}$;

Isotropic (t-r) solution: $\quad \gamma_{ER}(r) \sim \frac{1}{4\pi\rho_G^2}\left(\frac{x}{1-x}\right)^{1/2}$;

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$\quad$ solution: $\quad \gamma_{PGM}(r) \sim \frac{1}{4\pi\rho_G^2}\left(\frac{r}{\rho_G}\right)^2 e^{\rho_G/2r}$.

As we can see:

1) Most of the listed solutions are physically different. Only two of them: Schwarzschild and Einstein-Rosen ones, have the same singularity at the place of the point source and, as a result, have diffeomorphic spaces $\mathcal{M}^{(3)}\{g_{mn}(\mathbf{r})\}$, which can be related by a regular rho-gauge transformation.

2) As a result of the alteration of the physical meaning of the variable $\rho = r$ inside the sphere of radius ρ_H, in Hilbert gauge the coefficient $\gamma(r)$ tends to *imaginary* infinity for $r \to 0$. This is in a sharp contrast to the *real* asymptotic of all other solutions in the limit $r \to 0$.

3) The Droste solution is a regular one and corresponds to a gravitational mass defect with $\varrho_D = \left(\sqrt{5}-1\right)/2 \approx 0.61803$. The golden-ratio-conjugate number $\left(\sqrt{5}-1\right)/2$, called sometimes also "a silver ratio", appears in our problem as a root of the equation $1/\varrho - \varrho = 1$, which belongs to the interval $\varrho \in (0,1)$. (The other root of the same equation is $-\left(\sqrt{5}+1\right)/2 < 0$ and does not correspond to a physical solution.) Hence, the Droste solution can be transformed to the form (22), or (26) by a proper regular rho-gauge transformation.

4) The Weyl solution resembles a regular solution with $\varrho_W = 1$ and $\gamma_W(0) = 0$. Actually the exact invariant δ-function for this solution may be written in the form (see the Appendix B):

$$\delta_{g,W}(r) = \frac{16}{\pi\rho_G^2}\left(\left(\frac{r}{\rho_G}\right)^4 \delta(r) + \left(\frac{\rho_G}{r}\right)^4 \delta\left(\frac{\rho_G^2}{r}\right)\right). \quad (36)$$

This distribution acts as a zero-functional on the standard class of smooth test functions with compact support, which are finite at the points $r = 0, \infty$.

The formula (36) shows that:

a) The Weyl solution describes a problem with a spherical symmetry in the presence of *two* point sources: one with $\varrho_W^0 = 1$ at the point $r = 0$ and another one with

$\varrho_W^\infty = 1$ at the point $r = \infty$. Hence, it turns out that this solution is the first *exact* analytical two-particle-like solution of Einstein equations (1). Unfortunately, as a two-point solution, the Weyl solution is physically trivial: it does not describe the dynamics of two point particles at a finite distance, but rather gives only a static state of two particles at an infinite distance.

b) To make these statements correct, one has to compactify the comformally flat 3D space $\mathcal{M}^{(3)}\{-g_{mn}(\mathbf{r})\}$ joining it to the infinite point $r = \infty$. Then both the Weyl solution *and* its source will be invariant with respect to the inversion $r \to \rho_G^2/r$.

5) The isotropic (t-r) solution resembles a regular one, but in it $x \approx 1.27846$ is the only real root of the equation $x + \ln(x-1) = 0$ and gives a non-physical value of the parameter $\varrho_I \approx 2.14269$, which does not belong to the physical interval $(0,1)$.

6) The singularity of the coefficient $\gamma(r)$ in front of the usual 1D Dirac δ-function in the Pugachev-Gun'ko-Menzel solution is stronger than any polynomial ones in the other listed solutions.

B. Local Geometrical Singularities of the Regular Static Spherically Symmetric Space-Times

It is well known that the scalar invariants of the Riemann tensor allow a manifestly coordinate independent description of the geometry of space-time manifold. The problem of a single point particle can be considered as an extreme case of a perfect fluid, which consists of only one particle. According to the article [24], the maximal number of independent *real* invariants for such a fluid is 9. These are the scalar curvature R and the standard invariants r_1, w_1, w_2, m_3, m_5. (See [24] for their definitions.) The invariants R, r_1 and m_3 are real numbers and the invariants w_1, w_2 and m_5 are, in general, complex numbers. The form of these invariants for spherically symmetric space-times in normal field variables is presented in Appendix C. Using these formulas we obtain for the regular solutions (22) in the basic gauge the following simple invariants:

$$I_1 = \frac{1}{8\rho_G^2}\frac{(1-\varrho^2)^4}{\varrho^2}\delta\left(\frac{r}{\rho_G}\right) = \frac{\pi(1-\varrho^2)^3}{2\varrho}\delta_g\left(\frac{r}{\rho_G}\right) = -\frac{1}{2}R(r),$$

$$I_2 = 0,$$

$$I_3 = \frac{\Theta(r/\rho_G)-1}{8\rho_G^2\varrho^2}\left(1-\varrho^2 e^{4r/\rho_G}\right)^4 = \frac{\rho_G^2}{8\varrho^2}\frac{\Theta(r/\rho_G)-1}{\rho(r)^4},$$

$$I_4 = \frac{\Theta(r/\rho_G)}{4\rho_G^2}\left(1-\varrho^2 e^{4r/\rho_G}\right)^3 = \frac{\rho_G}{4}\frac{\Theta(r/\rho_G)}{\rho(r)^3}. \quad (37)$$

As can be seen easily:

1) The invariants $I_{1,...,4}$ of the Riemann tensor are well defined distributions. This is in accordance with the general expectations, described in the articles [25], where one can find a correct mathematical treatment of distributional valued curvature tensors in GR.

As we saw, the manifold $\mathcal{M}^{(1,3)}\{g_{\mu\nu}(x)\}$ for our regular solutions has a geometrical singularity at the point, where the physical massive point particle is placed.

Note that the metric for the regular solutions is globally continuous, but its first derivative with respect to the radial variable r has a finite jump at the point source. This jump is needed for a correct mathematical description of the delta-function distribution of matter of the massive point source in the right hand side of Einstein equations (1).

During the calculation of the expressions $I_{3,4}$ we have used once more our assumption $\Theta(0) = 1$.

Three of the invariants (37) are independent on the real axes $r \in (-\infty, \infty)$ and this is the true number of the independent invariants in the problem of a single point source of gravity. On the real physical interval $r \in [0, \infty)$ one has $I_3 = 0$ and we remain with only two independent invariants. For $r \in (0, \infty)$ the only independent invariant is I_4, as is well known from the case of Hilbert solution.

2) The scalar invariants $I_{1,\dots,4}$ have the same form as in Eq. (37) for the regular solutions in the representation (26), obtained via the diffeomorphic mapping (25).

3) All other geometrical invariants $r_1(r)$, $w_{1,2}(r)$, $m_{3,5}(r)$ include degrees of Dirac δ-function and are not well defined distributions. Therefore the choice of the simple invariants $I_{1,\dots,4}$ is essential and allows us to use solutions of Einstein equations, which are distributions.

To the best of authors knowledge, the requirement to make possible the correct use of the theory of distributions is a novel criteria for the choice of invariants of the curvature tensor and seems to not have been used until now. It is curious to know if it is possible to apply this new criteria in more general cases as opposed to the case of static spherically symmetric space-times.

4) The geometry of the space-time depends essentially on both parameters M and ϱ, which define the regular solutions (26) of Einstein equations (1). Hence, for different values of the two parameters these solutions describe non-diffeomorphic space-times with different geometry.

C. Event Horizon for Static Spherically Symmetric Space-Times with Point Singularity

According to the well known theorems by Hawking, Penrose, Israel and many other investigators, the only solution with a regular event horizon not only in GR, but in the theories with scalar field(s) and in more general theories of gravity, is the Hilbert one [12]. As we have seen, this solution has a pure geometrical nature and does not describe a gravitational field of a point particle with bare mechanical mass $M_0 \neq 0$. The strong mathematical results like the well known no hear theorems, etc, for the case of metrics with an event horizon can be shown to be based on to the assumption that such horizon is *indeed present* in the solution. These mathematical results do not contradict to the ones, obtained in the present article, because the other solutions, which we have considered together with the Hilbert one, do not have an event horizon at all.

Indeed, for the point ρ_H at which $g_{tt}(\rho_H) = 0$ one obtains $\rho(\rho_H) = \rho_G$. The last equations do not have any solution $\rho_H \in \mathcal{C}^{(1)}$ for the regular solutions (22), (26). The absence of a horizon in the physical domain $r \in [0, \infty)$ is obvious in the representation (29) of the regular solutions.

For the other classical solutions one obtains in the limit $r \to \rho_H$ as follows:

Schwarzschild solution:	$g_{rr}(\rho_H) \sim -\frac{\rho_G}{r}$,	$\rho_H = 0$;
Hilbert solution:	$g_{rr}(\rho_H) \sim -\frac{\rho_G}{r - \rho_G}$,	$\rho_H = \rho_G$;
Droste solution:	$g_{rr}(\rho_H) \sim -1$,	$\rho_H = 0$;
Weyl solution:	$g_{rr}(\rho_H) \sim -1$,	$\rho_H = \rho_G$;
Einstein-Rosen solution:	$g_{rr}(\rho_H) \sim -\frac{\rho_G}{r}$,	$\rho_H = 0$;
Isotropic (t-r) solution:	$g_{rr}(\rho_H) \sim e^{\frac{r}{\rho_G}}$,	$\rho_H = -\infty$;

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solution: $g_{rr}(r) \sim -\left(\frac{\rho_G}{r}\right)^4 e^{-\frac{r}{\rho_G}}$, $\rho_H = 0$.

As one can see, only for Hilbert, Schwarzschild and Einstein-Rosen solutions the metric component $g_{rr}(r)$ has a simple pole at the point ρ_H. In the last two cases this is a point in the corresponding manifold $\mathcal{M}^{(3)}\{g_{mn}(\mathbf{r})\}$, where the center of symmetry is placed, not a horizon.

X. ON THE GEOMETRY OF MASSIVE POINT SOURCE IN GR

It is curious that for a metric, given by a regular solution, the 3D-volume of a ball with a small radius $r_b < \rho_G$, centered at the source, is

$$Vol(r_b) = \frac{4}{3}\pi\rho_G^3 \frac{12\varrho}{(1 - \varrho^2)^4}\frac{r_b}{\rho_G} + \mathcal{O}_2(\frac{r_b}{\rho_G}).$$

It goes to zero linearly with respect to $r_b \to 0$, in contrast to the Euclidean case, where $Vol_E(r_b) \sim r_b^3$. This happens, because for the regular solutions (22), (26) we obtain $\sqrt{|^3g|} = \rho(r)^2 \sin\theta \to \rho_0 \sin\theta \neq 0$ in the limit $r \to 0$. Nevertheless, $lim_{r_b \to 0} Vol(r_b) = 0$ and this legitimates the use of the term "a point source of gravity" in the problem at hand: the source can be surrounded by a sphere with an arbitrary small volume $Vol(r_b)$ in it and with an arbitrary small radius r_b.

In contrast, when $r_b \to 0$ the area of the ball's surface has a finite limit: $\frac{4\pi\rho_G^2}{(1 - \varrho^2)^2} > 4\pi\rho_G^2$, and the radii of the big circles on this surface tend to a finite number $\frac{2\pi\rho_G}{1 - \varrho^2} > 2\pi\rho_G$.

Such unusual geometry, created by the massive point sources in GR, may have an interesting physical consequences. For example, space-times, defined by the regular solutions (22), (26), have an unique property: When one approaches the point source, its luminosity remains finite. This leads to a very important modification of Gauss theorem in the corresponding 3D spaces.

After all, this modification may solve the well known problem of the *classical* divergences in field theory, because, as we see, GR offers a natural cut-off parameter $\tilde{r} = \rho_G / \ln(1/\varrho^2)$ for fields, created by massive point particles. If this will be confirmed by more detailed calculations, the price, one must pay for the possibility to overcome the old classical divergences problem will be to accept the idea, that the point objects can have a nonzero surface, having at the same time a zero volume and a zero size r.

On the other hand, as we have seen, the inclusion of point particles in GR is impossible without such unusual geometry. The Hilbert solution does not offer any more attractive alternative for description of a source of gravity as a physical massive *point*, placed at the coordinate point $\rho = 0$, and with *usual* properties of a geometrical point. Hence, in GR we have no possibility to introduce a notion of a physical point particle with nonzero bare mass without a proper change of the standard definition for point particle as a mathematical object, which has simultaneously zero size, zero surface and zero volume. We are forced to remove at least one of these three usual requirements as a consequence of the concentration of the bare mass M_0 at only one 3D space point.

From physical point of view the new definition for physical point particle, at which we arrived in this article, is obviously preferable.

Moreover, it seems natural to have an essential difference between the geometry of the "empty" space-time points and that of the matter points in GR. The same geometry for such essentially different objects is possible only in the classical physics, where the space-time geometry does not depend on the matter. Only under the last assumption there is no *geometrical* difference between pure mathematical space points and matter points. The intriguing new situation, described in this article, deserves further careful analysis.

The appearance of the above nonstandard geometry in the point mass problem in GR was pointed at first by Marcel Brillouin [27]. Our consideration of the regular solutions (22), (26) confirms his point of view on the character of the singularity of the gravitational field of massive point particles in GR.

XI. CONCLUDING REMARKS

The most important result of the present article is the explicit indication of the fact that there exist infinitely many different static solutions of Einstein equations (1) with spherical symmetry, a point singularity, placed at the center of symmetry, and with the same Keplerian mass M. These solutions fall into different gauge classes, which describe physically and geometrically different space-times. Some of them were discovered at the early stage of development of GR, but up to now they are often considered as equivalent representations of some "unique" solution

which depends on only one parameter – the Keplerian mass M. As shown in Section IX A, this is not the case.

As a consequence of Birkhoff theorem, when expressed in the same variables, the solutions with the same mass M indeed coincide in their common regular domain of coexistence – outside the singularities. In contrast, they may have a different behavior at the corresponding singular points.

A correct description of a massive point source of gravity is impossible, making use of most of these classical solutions.

Using novel normal coordinates for the gravitational field of a single point particle with bare mechanical mass M_0 we are able to describe correctly the massive point source of gravity in GR. The singular gauge transformations yield the possibility to overcome the restriction to have a zero bare mass M_0 for neutral point particles in GR [19]. It turns out that this problem has a two-parametric family of regular solutions.

One of the parameters – the bare mass M_0 of the point source, can be obtained in a form of surface integral, integrating both sides of Eq. (34) on the whole 3D space $\mathcal{M}^{(3)}\{-g_{mn}(\mathbf{r})\}$:

$$M_0 = \frac{1}{4\pi G_N} \int_{\mathcal{M}^3} d^3\mathbf{r} \sqrt{|^3g|} \, \Delta \left(\ln \sqrt{g_{tt}} \right) =$$
$$\frac{1}{4\pi G_N} \oint_{\partial \mathcal{M}^3} d^2\sigma_i \sqrt{|g|} \, g^{ij} \, \partial_j \sqrt{g^{tt}}.$$

The second parameter – the Keplerian mass M, is described, as shown in Section VIII, by the total energy:

$$M = \int_{\mathcal{M}^3} d^3\mathbf{r} \sqrt{|^3g|} \, \mathcal{H}_{tot} = \int_{\mathcal{M}^3} d^3\mathbf{r} \sqrt{|g|} \, M_0 \, \delta_g^{(3)}(\mathbf{r}).$$

These two parameters define the *gauge class* of the given regular solution. Obviously the parameters M and M_0 are invariant under regular static gauge transformations, i.e., under diffeomorphisms of the 3D space $\mathcal{M}^{(3)}\{-g_{mn}(\mathbf{r})\}$. It is an analytical manifold with a strong singularity at the place of the massive point particle. For every regular solution both parameters are finite and positive and satisfy the additional physical requirement $0 < M < M_0$.

It is convenient to use a more physical set of continuous parameters for fixing the regular solutions, namely: the Keplerian mass $M \in (0, \infty)$ and the gravitational mass defect ratio $\varrho = \frac{M}{M_0} \in (0, 1)$.

The only classical solution, which is regular, is the Droste one. For this solution the gravitational mass defect ratio is $\varrho_D = \left(\sqrt{5} - 1 \right)/2$.

For the regular solutions the physical values of the optical luminosity distance ρ are in the semi-constraint interval $\rho \in [\frac{\rho_G}{1-\varrho^2}, \infty)$.

Outside the source, i.e., for $\rho > \frac{\rho_G}{1-\varrho^2}$, the Birkhoff theorem is strictly respected for all regular solutions.

The metric, defined by a regular solution, is *globally* continuous, but its first derivatives have a jump at the

point source. Our explicit results justify the Raju's conclusion [25] that GR will be consistent with the existence of point particles if one assumes the metric to be at most of class C^0 and show that the Hawking-Penrose singularity theory, based on the assumption that the components of $g_{\mu\nu}$ are *at least* C^1 functions, must be reconsidered.

For the class of regular solutions, written in the form (22), (26), or (29), the non-physical interval of the optical luminosity distance $\rho \in [0, \frac{\rho_G}{1-\varrho^2}]$, which includes the luminosity radius $\rho_H = \rho_G$, can be considered as an "optical illusion". All pure mathematical objects, which belong to this interval, are in the imaginary domain "behind" the real physical source of the gravitational field and have to be considered as a specific kind of optical "mirage". This is in agreement with Dirac's conclusion about the non-physical character of the inner domain $\rho \leq \rho_G$ for the Hilbert solution [26] and extends this conclusion on the whole non-physical interval of the optical luminosity distance $\rho \in [0, \frac{\rho_G}{1-\varrho^2}] > \rho_H$ for a given *regular* solution.

We are forced to cut the Hilbert form of the Schwarzschild solution at the value $\rho_0 = \frac{\rho_G}{1-\varrho^2}$ because of the presence of the matter point mass. Its presence ultimately requires a definite jump of the first derivatives of the metric. The jump is needed to make the Einstein tensor coherent with the energy-momentum stress tensor of the point particle. This cutting can be considered as a further development of Dirac's idea [26]:

"Each particle must have a finite size no smaller than the Schwarzschild radius. I tried for some time to work with a particle with radius equal to the Schwarzschild radius, but I found great difficulties, because the field at the Schwarzschild radius is so strongly singular, and it seems that a more profitable line of investigation is to take a particle bigger than the Schwarzschild radius and to try to construct a theory for such particle interacting with gravitational field."

For a precise understanding of these statements one has to take into account that the above Dirac's idea is expressed in terms of the luminosity distance ρ.

An alternative development, which is much closer to Dirac approach to the above problem, can be found in the recent articles [28].

The geometry around physical massive point particles is essentially different from the geometry around the "empty" geometrical points. This unusual geometry, described at first by Brillouin, may offer a new way for overcoming of the *classical* fields divergence problems. The new possibility differs from the one, used in [19] for charged point particles. It is based on the existence of a natural cut-off parameter $\tilde{r} = G_N M \left(c^2 \ln\left(\frac{M_0}{M}\right)\right)^{-1}$ and on a new interpretation of the relation between the singular mass distribution of a point particle and the geometry of the space-time around this particle.

According to the remarkable comment by Poincaré [29] real problems can never be considered as solved or unsolved, but rather they are always *more or less* solved.

The strong *physical* singularity at the "event horizon"

of the Hilbert solution, stressed by Dirac and further physical consequences, which one can derive from the new regular solutions (26) of the old problem, considered here will be discussed in a separate article. In this one we will add some more remarks.

Our consideration shows that the observed by the astronomers compact dark objects (CDO), called by them black *holes* without any direct evidences for the existence of *real* event horizons [30], can not be described theoretically by solutions (8) in Hilbert gauge, if we assume that these objects are made from matter of some nonzero bare mechanical mass $M_0 \neq 0$. At present the only real fact is the existence of a massive invisible CDO with Keplerian mass M, which is too large with respect to the conventional understanding of the stars physics.

Concerning these unusual CDO, most probably one actually has to solve a much more general problem. Since a significant amount of invisible dark matter, which manifests itself only due to its gravitational field, is observed in the Nature at very different scales: in the clusters of galaxies, in the halos of the galaxies, at the center of our galaxy, and as a compact dark components in some binary star systems, one is tempted to look for some universal explanation of all these phenomena. Obviously such universal explanation can not be based on Schwarzschild solution in Hilbert gauge and one has to look for some other theoretical approach. A similar idea was pointed out independently in the recent articles [31].

At the end we wish to mention some additional open problems, both mathematical and physical ones, connected with the gravitational field of point particles.

Our consideration was essentially restricted to the real domain of variables. The only exception was the description of the singular change (28) of radial variable. It is well known that the natural domain for study of the solutions of holomorphic differential equations is the complex one. A complete knowledge of the solutions of such equations is impossible without description of all singularities of the solutions in the complex domain. Therefore, looking for a complete analysis of the solutions of some differential equation of n-th order $f\left(w^{[n]}(z), w^{[n-1]}(z), ..., w^{[1]}(z), w(z), z\right) = 0$ for a function $w(z), z \in \mathcal{C}^{(1)}$, where $f(...)$ is a holomorphic function of the corresponding complex variables, one has to consider these solutions as a holomorphic functions of complex variable z. Now the most important issue becomes the study of the singular points of the solutions in the *whole* complex domain.

For the Einstein's equations (1) such a *four-dimensional* complex analysis is impossible at present, because the corresponding mathematical methods are not developed enough. But for the problem of single point source in GR, in its 1D formulation, used in this article, one can use the classical complex analysis. Moreover, using the well known complex representation of distributions (see Bremermann in [8]), it is not difficult to generalize the classical results for the case when the differential equation has on the right hand side a distri-

bution, which depends on the variable z. Such a term may lead to a discontinuity of the solution $w(z)$, or of its derivatives. The corresponding complex analysis of the solution remains an important open mathematical problem, together with the problem of finding and classifying *all* static solutions of Einstein equations with one point singularity and spherical symmetry.

The most important physical problem becomes to find criteria for an experimental and/or observational probe of different solutions of Einstein equations with spherical symmetry. Since there exist *physical and geometrical* differences between the solutions of this type, one can find a real way of distinguishing them experimentally. Especially, a problem of the present day is to answer the question about which one of the spherically symmetric solution gives the right description of the observed CDO, if one hopes that the CDO can be considered in a good approximation as spherically symmetric objects with some Keplerian mass M and some mass defect ratio $\varrho \in (0,1)$. In this case, according to the Birkhoff theorem, we must use the new solutions (29) for description of their gravitational field in the outer vacuum domain, outside the CDO.

On the same reason the solutions (29) must be used, too, for description of the vacuum gravitational field of spherically symmetric bodies of finite size, made of usual matter, outside their radius R. Because of the specific character of the correction of the Newton's low of gravitation, defined by the term $G_{_N} M c^{-2}/\ln\left(\frac{M_0}{M}\right)$ in the potential $\varphi_{_G}(r; M, M_0)$ (30), the difference between our solutions (29) and the Hilbert one is in higher order relativistic terms, when $\ln\left(\frac{M_0}{M}\right) \sim 1$. This difference will not influence the rho-gauge invariant local effects, like perihelion shift, deviation of light rays, time-dilation of electromagnetic pulses, and so on, but it may be essential for the quantities, which depend on the precise form of the potential $\varphi_{_G}(r; M, M_0)$ (30). For all bodies in the solar system the term $G_{_N} M c^{-2}/\ln\left(\frac{M_0}{M}\right)$ is very small in comparison with their radius R: $(\rho_{_G}/2R)_{Earth} \sim 10^{-10}$, $(\rho_{_G}/2R)_{\odot} \sim 10^{-6}$, etc. Nevertheless, the small differences between the Newton's potential and the modified one may be observed in the near future in the extremely high precision measurements under programs like APOLLO, LATOR, etc. (see for example [32]), if we will be able to find a proper quantities, which depend on the precise form of the potential $\varphi_{_G}(r; M, M_0)$ (30). A better possibility for observation of a deviations from Newton's gravitational potential and in four-interval may offer neutron stars, since for them $(\rho_{_G}/2R)_{n*} \sim 0.17 - 0.35$.

The results of the present article may have an important impact on the problem of gravitational collapse in GR, too. Up to now most of the known to the author investigations presuppose to have as a final state of the collapsing object the Hilbert solution, or a proper generalization of it with some kind of event horizon, in the spirit of the cosmic censorship hypothesis. In the last decade more attention was paid to the final states with a necked singularities. The existence of a two-parametric class of regular static solutions without event horizon opens a new perspective for these investigations. Most probably the proper understanding of the origin of the mass defect of point particle, introduced in the present article, will be reached by a correct description of the gravitational collapse.

Without any doubts, the inclusion of the new static spherically symmetric regular solutions of Einstein equations in the corresponding investigations will open new perspectives for a further developments in GR, as well as in its modern generalizations.

APPENDIX A: DIRECT DERIVATION OF FIELD EQUATIONS FROM EINSTEIN EQUATIONS

Using the normal field variables, introduced in Section V, one can write down the nonzero mixed components G^μ_ν of Einstein tensor in the basic regular gauge ($\bar\varphi(r) \equiv 0$) as follows:

$$G^t_t = \frac{e^{2\varphi_1 - 4\varphi_2}}{\bar\rho^2}\left(2\bar\rho^2 \varphi_1'' - 2\bar\rho^2 \varphi_2'' - (\bar\rho\varphi_1')^2 + (\bar\rho\varphi_2')^2 + e^{2\varphi_2}\right),$$

$$G^r_r = \frac{e^{2\varphi_1 - 4\varphi_2}}{\bar\rho^2}\left((\bar\rho\varphi_1')^2 - (\bar\rho\varphi_2')^2 + e^{2\varphi_2}\right),$$

$$G^\theta_\theta = G^\phi_\phi = -\frac{e^{2\varphi_1 - 4\varphi_2}}{\bar\rho^2}\left(\bar\rho^2\varphi_2'' + (\bar\rho\varphi_1')^2 - (\bar\rho\varphi_2')^2\right). \qquad (A1)$$

Taking into account that the only nonzero component of the energy-momentum tensor is T^t_t and combining the Einstein equations (1), one immediately obtains the field equations (13) and the constraint (18). A slightly more

general consideration with an arbitrary rho-gauge fixing function $\bar{\varphi}(r)$ yields the equations (11) and the constraint(12).

A similar derivation produces the equations (5), which present another version of the field equations in spherically symmetrical space-times. In this case:

$$G_t^t = \frac{1}{-g_{rr}}\left(-2\left(\frac{\rho'}{\rho}\right)' - 3\left(\frac{\rho'}{\rho}\right)^2 + 2\frac{\rho'}{\rho}\frac{\sqrt{-g_{rr}}'}{\sqrt{-g_{rr}}}\right) + \frac{1}{\rho^2},$$

$$G_r^r = \frac{1}{-g_{rr}}\left(-\left(\frac{\rho'}{\rho}\right)^2 + 2\frac{\rho'}{\rho}\frac{\sqrt{g_{tt}}'}{\sqrt{g_{tt}}}\right) + \frac{1}{\rho^2}, \quad \text{(A2)}$$

$$G_\theta^\theta = G_\phi^\phi = \frac{1}{-g_{rr}}\left(-\left(\frac{\rho'}{\rho}\right)' - \left(\frac{\rho'}{\rho}\right)^2 - \frac{\rho'}{\rho}\frac{\sqrt{g_{tt}}'}{\sqrt{g_{tt}}} + \right.$$

$$\left. \frac{\rho'}{\rho}\frac{\sqrt{-g_{rr}}'}{\sqrt{-g_{rr}}} - \left(\frac{\sqrt{g_{tt}}'}{\sqrt{g_{tt}}}\right)' - \left(\frac{\sqrt{g_{tt}}'}{\sqrt{g_{tt}}}\right)^2 + \frac{\sqrt{g_{tt}}'}{\sqrt{g_{tt}}}\frac{\sqrt{-g_{rr}}'}{\sqrt{-g_{rr}}}\right).$$

APPENDIX B: INVARIANT DELTA FUNCTION FOR WEYL SOLUTION

The function $\rho_W(r) = \frac{1}{4}\left(\sqrt{r/\rho_G} + \sqrt{\rho_G/r}\right)^2 \geq \rho_G$ which describes the Weyl rho-gauge fixing is an one-valued function of $r \in (0,\infty)$, but its inverse function $r_W(\rho)$ is a two-valued one: $r_W(\rho) = r_W^\pm(\rho) = \rho\left(1 \pm \sqrt{1 - \rho_G/\rho}\right)^2$, $\rho \in [\rho_G,\infty)$. Obtaining information about distant objects only in optical way, i.e. by measuring only the luminosity distance ρ, one is not able to choose between the two branches of this function and therefore one has to consider both of them as possible results of the observations. Hence, the 3D space $\mathcal{M}_W^{(3)}\{g_{mn}(\mathbf{r})\}$, which appears in the Weyl gauge, has to be considered as a two-sheeted Riemann surface with a branch point at the 2D sphere $\rho^* = \rho_G$ in $\mathcal{M}_W^{(3)}\{g_{\tilde{m}\tilde{n}}(\rho,\tilde{\theta},\tilde{\phi})\}$ – the "observable" space with new spherical coordinates $\rho,\tilde{\theta},\tilde{\phi}$. Because of the limits $r_W^+(\rho) \to \infty$ and $r_W^-(\rho) \to 0$ for $\rho \to \infty$, one can write down the invariant δ-function in the case of Weyl gauge in the form

$$\delta_{g,W}(\rho) = \frac{\sqrt{1 - \rho_G/\rho}}{4\pi\rho^4}\,\delta\left(\frac{1}{\rho}\right). \quad \text{(B1)}$$

Now, taking into account that the equation $1/\rho(r) = 0$ has two solutions: $r = 0$ and $1/r = 0$ and using the standard formula for expansion of the distribution $\delta\left(1/\rho(r)\right)$, we obtain easily the representation (36).

APPENDIX C: INDEPENDENT NONZERO INVARIANTS OF RIEMANN CURVATURE IN NORMAL FIELD COORDINATES

Using the the representation (9) of the metric in normal field variables after some algebraic manipulations one

obtains the following expressions for the possibly independent nonzero invariants of Riemann curvature tensor in the problem at hand:

$$R = -2e^{2(\varphi_1 - 2\varphi_2)}\left(\varphi_1'' - 2E_2 - E_3\right),$$

$$r_1 = \frac{1}{4}e^{4(\varphi_1 - 2\varphi_2)}\left(2\left(\varphi_1'' - E_2 - E_3\right)^2 + \left(\varphi_1'' + E_3\right)^2\right),$$

$$w_1 = \frac{C^2}{6}, \quad w_2 = -\frac{C^3}{36}, \quad \text{(C1)}$$

$$m_3 = \frac{C^2}{12}e^{4(\varphi_1 - 2\varphi_2)}\left(\left(\varphi_1'' - E_2 - E_3\right)^2 + 2\left(\varphi_1'' + E_3\right)^2\right),$$

$$m_5 = \frac{C^3}{36}e^{4(\varphi_1 - 2\varphi_2)}\left(\frac{1}{2}\left(\varphi_1'' - E_2 - E_3\right)^2 - 2\left(\varphi_1'' + E_3\right)^2\right),$$

where $E_2 := \varphi_2'' - e^{2\varphi_2}/\bar{\rho}^2$ and $E_3 := \varphi_1'^2 - \varphi_2'^2 + e^{2\varphi_2}/\bar{\rho}^2$.

As we see, the invariants $w_{1,2}$ and m_5 in our problem are real. Hence, in it we may have at most 6 independent invariants.

In addition $(w_1)^3 = 6\,(w_2)^2$ and the metric (9) falls into the class II of Petrov classification [24]. As a result the number of the independent invariants is at most 5.

For our purposes it is more suitable to use as an independent invariant

$$C := -e^{2\varphi_1 - 4\varphi_2}\left(2\varphi_1'' + 6\varphi_1'\left(\varphi_1' - \varphi_2'\right) - E_2 - 2E_3\right) \quad \text{(C2)}$$

instead of the invariants w_1 and w_2. The invariant C has the following advantages:

i) It is linear with respect of the derivative φ_1''. This property makes meaningful the expression (C2) in the cases when φ_1'' is a distribution;

ii) It is a homogeneous function of first degree with respect to the Weyl conformal tensor $C_{\mu\nu,\lambda}{}^\xi$;

iii) It is proportional to an eigenvalue of a proper tensor, as pointed, for example, in Landau and Lifshitz [1].

iv) The difference between the scalar curvature R and the invariant (C2) yields a new invariant

$$D := R - C = 3\,e^{2\varphi_1 - 4\varphi_2}\left(2\varphi_1'\left(\varphi_1' - \varphi_2'\right) + E_2\right) \quad \text{(C3)}$$

which does not include the derivative φ_1''. Due to this property the invariant D in the case of regular solutions (22), (26) will not contain a Dirac δ-function. One can use the new invariant D as an independent one, instead of the invariant C.

As seen from Eq. (C1), the following functions of the invariants r_1, m_3 and m_5:

$$e^{2(\varphi_1 - 2\varphi_2)}\left(\varphi_1'' + E_3\right) = \sqrt{4\left(m_3/w_1 - r_1\right)/3},$$

$$e^{2(\varphi_1 - 2\varphi_2)}\left(2\varphi_1'' - E_2 - E_3\right) = \sqrt{2\left(4r_1 - m_3/w_1\right)/3} \quad \text{(C4)}$$

are linear with respect to the derivative φ_1''. Using proper linear combinations of these invariants and the scalar

curvature R one can easily check that the quantities $E_2\, e^{2(\varphi_1-2\varphi_2)}$, and $E_3\, e^{2(\varphi_1-2\varphi_2)}$ are independent invariants too.

Thus, our final result is that, in general, for the metric (9) in the basic regular gauge $\bar\varphi \equiv 0$ the Riemann curvature tensor has the following four independent invariants:

$$I_1 := e^{2(\varphi_1-2\varphi_2)}\varphi_1'',$$
$$I_2 := e^{2(\varphi_1-2\varphi_2)}\left(\varphi_2'' - e^{2\varphi_2}/\bar\rho^2\right), \qquad \text{(C5)}$$

$$I_3 := e^{2(\varphi_1-2\varphi_2)}\left(-\varphi_1'^{\,2} + \varphi_2'^{\,2} - e^{2\varphi_2}/\bar\rho^2\right)/2,$$
$$I_4 := e^{2(\varphi_1-2\varphi_2)}\varphi_1'\left(\varphi_1' - \varphi_2'\right)/2.$$

These invariants are linear with respect to the second derivatives of the functions $\varphi_{1,2}$ – a property, which is of critical importance when we have to work with distributions $\varphi_{1,2}''$.

[1] L. D. Landau, E. M. Lifshitz, *The Classical Theory of Fields*, 2d ed.; Reading, Mass: Addison-Wesley, 1962; V. A. Fock, *The Theory of Space, Time and Gravitation*, Pergamon, Oxford, 1964. R. C. Tolman, *Reativity, Thermodynamics and Cosmology*, Claderon Press, Oxford, 1969. S. Weinberg,*Gravitation and Cosmology*, Wiley, N.Y., 1972; C. Misner, K. S. Thorne, J. A. Wheeler, *Gravity*, W. H. Freemand&Co.,1973.

[2] D. Kramer, H. Stephani, M. Maccallum, E. Herlt, Ed. E. Schmutzer, *Exact Solutions of the Einstein Equations*, Deutscher Verlag der Wissenschaften, Berlin, 1980. H. Stephani, D. Kramer, M. Maccallum, E. Herlt, *Exact Solutions of Einstein's Field Equations*, Sec. Ed., Cambridge University Press, 2003.

[3] M. Göckeler, T. Schücker, *Differential Geometry, Gauge Theories, and Gravity*, Cambridge University Press, Cambridge, 1987.

[4] V. Rubakov, *Classical Theory of Gauge Fields*, Princeton University Press, Princeton and Oxford, 1999.

[5] K. Schwarzschild, Sitzungsber. Preus. Acad. Wiss. Phys. Math. Kl., 189 (1916).

[6] A. S. Eddington, *The mathematical theory of relativity*, 2nd ed. Cambridge, University Press, 1930 (repr.1963).

[7] R. Gautreau, B. Hoffmann, Phys. Rev. D**17**, 2552 (1978).

[8] L. Schwartz, *Théorie des distributions* I, II, Paris, 1950-51; I. M. Gel'fand, G. E. Shilov, *Generalized Finctions*, N.Y., Academic Press, 1964; H. Bremermann, *Distrinutions, Complex Variables and Fourier Transform*, Addison-Wesley Publ. Co. Reading, Massachusetts, 1965.

[9] J. T. Combridge, Phil. Mag. **45**, 726 (1923); H, Janne, Bull. Acad. R. Belg. **9**, 484 (1923). F.J.Belinfante, Phys. Rev. **98**, 793 (1955). Ya. I. Pugachev and V. D. Gun'ko, Soviet Phys. Journ., Izvestiya Vysshikh Uchebnykh Zavednii, Fizika, **10**, 46, (1974).

[10] D. Hilbert, Nachr. Ges. Wiss. Götingen, Math. Phys. Kl., 53 (1917).

[11] S. Antoci, D.-E. Liebscher, Astron. Nachr. **322**, 137 (2001); S. Antoci, D.-E. Liebscher, L. Mihich, Class. Quant. Grav. **18**, 3466 (2001).

[12] V. P. Frolov, I. D. Novikov, *Black Hole Physics*, Kluwer Acad. Publ., 1998.

[13] J.Droste, Proc.K.Ned.Acad.Wet., Ser., A**19**, 197 (1917).

[14] H. Weyl, Ann. Phys., Leipzig, **54**, 117 (1917).

[15] A. Einstein, N. Rosen, Phys. Rev., **48**, 73 (1935).

[16] D. H. Menzel, Mém. Soc. Roy. Sci. Liége, **9**, 343 (1976).

[17] Ll. Bel, J. Math. Phys. **10**, 1501 (1969); Gen. Rel. and Grav. **1**, 337 (1971); Ll. Bel., J. Llosa, Gen. Rel. and Grav. **27**, 1089 (1995); Ll. Bel, Gen. Rel. and Grav. **28**, 1139 (1996);

[18] J. M. Aguirregabiria, Ll. Bel, J. Martin, A. Molina, E. Ruiz gr-qc/0104019, Gen. Rel. and Grav. **33** 1809 (2001); J. M. Aguirregabiria, Ll. Bel, gr-qc/0105043, Gen. Rel. and Grav. **33**, 2049 (2001) ; Ll. Bel, gr-qc/0210057.

[19] E. Arnowitt, S. Deser, C. M. Misner, Phys. Rev. Lett. D**4**, 375 (1960); Phys. Rev. **120** 321 (1960); in *Gravitation: an introduction to current research*, Ed. L. Witten, John Wiley, N.Y., 1962.

[20] A. Lichnerowicz, *Théories Relativistes de la Gravitation et de l'Electromagnétisme*, Paris, Masson, 1955.

[21] Y. Bruhat, Cauchy Problem, in *Gravitation: an introduction to current research*, ed. L. Witten, John Wiley, N.Y. , 1962.

[22] L. S. Abrams, Phys. Rev. D**20**, 2474 (1979); Can. J. Phys., **67**, 919 (1989); Physica **A**, 131 (1996); Int. J. Theor. Phys. **35**, 2661 (1996).

[23] R. Geroch, J. Traschen, Phys. Rev. D**36** 1017 (1987).

[24] J. Carminati, R. G. McLenaghan, J. Math. Phys. **32**, 3135 (1991).

[25] A. H. Taub, J. Math. Phys. **21** 1423 (1979). C. K. Raju, J. Phys. A: Math. Gen. **15** 1785 (1982).

[26] P. A. M.Dirac, Proc. Roy Soc. (London), A**270**, 354 (1962); Conference in Warszawa and Jablonna, L. Infeld ed., Gauthier-Villars, Paris (1964), pp. 163-175.

[27] M. Brillouin, Le Journal de physique et Le Radium **23**, 43 (1923).

[28] P. O. Mazur, E. Motolla, *Gravitational Condensate Stars: An Alternative to Black Holes*, gr-qc/0109035; M. Visser, D. L. Wiltshire, *Stable Gravistars - an Alternative to Black Holes*, gr-qc/0310107.

[29] H. Poincare, Science et Métode, Book 1, Flammarion, Paris, 1934.

[30] M. A. Abramowicz, W. Kluźniak, J-P. Lasota, Astron. Astrophys., **396**, L31 (2002); astro-ph/0207270.

[31] E. W. Mielke, F. E. Schunck, Nucl. Phys. B**564**, p.185 (2000); gr-qc/0001061. R. Narayan, *Evedences for the Black Hole Event Horison*, astro-ph/0310692.

[32] S. G. Turyshev, J. G. Williams, K. Nordtvedt, Jr., M. Shao, T. W. Murphy, Jr., *35 Years of Testing Relativistic Gravity: Where do we go from here?*, gr-qc/0311039.